

ADMISSIBLE LICHNEROWICZ COORDINATES FOR THE SCHWARZSCHILD METRIC

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Abstract

Global harmonic coordinates for the complete Schwarzschild metric are found for a more general case than that addressed in a previous work by Quan [4]. The supplementary constant that appears, in addition to the mass, is related to the stress quadrupole moment of a singular energy-momentum tensor. Similar calculations are also carried out in q-harmonic coordinates.

1. - Introduction

Since its advent, the theory of General Relativity has been used broadly to handle harmonic coordinates for several purposes. In many cases, such coordinates are used for reasons of mathematical comfort [1], while other times they have been used with the purpose of attributing them a special meaning [2]. In particular, they are considered to be a natural generalization of Euclidean Cartesian coordinates. Nevertheless, before recently no global system of harmonic and asymptotically Cartesian coordinates has been offered for a stellar-like model; that is, a system that is well defined everywhere (interior and exterior) and with a C^1 metric on the surface of the star (admissible coordinates in the sense of Lichnerowicz [3]). One example of this has been provided by Quan Hui Liu [4]: the complete Schwarzschild metric (interior and exterior) with $\mu_0 = 8/9$ (μ_0 being the dimensionless quotient between two times the mass of the star and its radius). In this case, it should be noticed that this special value of the μ_0 parameter leads to a simplification of the calculations but, unfortunately, it also provides a model of a star with divergences in the pressure at its center. This programme forces one to introduce two new constants (in addition to the mass) into the metric: one of them, Q_{ext} , at the exterior and another, Q_{int} , in the interior. These constants, which are defined with a length dimension in the above paper, prove to be proportional to the radius of the star, and according to the author are physically meaningless.

The aim of the present article is dual: on one hand, we shall develop Quan Hui Liu's programme but in a more realistic and general case for the μ_0 parameter ($\mu_0 < 8/9$). The solution of the problem for an arbitrary value of this parameter leads to a not well-known Heun differential equation [5] instead of the simple hypergeometric equation that appears

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in the limiting case $\mu_0 = 8/9$. On the other hand, by use of the linear approximation of vacuum and spherically symmetric Einstein equations we show that the exterior constant Q_{ext} is closely related to the stress quadrupole moment of the source.

In Section 2 the complete Schwarzschild metric is written in standard Droste coordinates simply to set the notation to be used, and the Darmois matching conditions [6] are briefly reviewed by writing the first and the second fundamental forms relative to the surface of the star.

In Section 3 the change of coordinates is found, from standard ones to general asymptotically Cartesian harmonic coordinates, two conditions being fulfilled: the new system of coordinates should preserve the spherical symmetry and diagonal structures of the metric (written in associated polar coordinates) and, in addition, the function relating both sets of coordinates must be C^1 on the surface of the star. These conditions allow us to express Quan's exterior and interior constants in terms of the change functions evaluated at the boundary. We then prove that the new coordinates are admissible coordinates in the sense of Lichnerowicz [3], i.e. the metric is C^1 on the surface.

Section 4 is devoted to interpreting the exterior constant, Q_{ext} . To do so, we first write the multipole expansion of the exterior Schwarzschild metric in the new set of harmonic coordinates found earlier (using the inverse of the radial coordinate, up to order five, as the parameter of the series). That expansion has already been shown, although in a slightly different way, by one of us [7]. Secondly, the more general solution of the vacuum Einstein equations with spherical symmetry in the linear approximation is made explicit and hence a suitable comparison with the previous expansion can be established, providing arguments for understanding the significance of the new constants appearing in the metric. Finally, it is checked that this solution also comes from some singular energy momentum tensor whose stress quadrupole moment proves to be "proportional" to the exterior constant of Quan (which means that Quan's constant is the factor multiplying the tensorial expression of that multipole moment).

In Section 5 we briefly discuss the results obtained for the same topic by using the so-called q-harmonic coordinates, a variety of harmonic coordinates introduced by L. Bel [8]. Most of the contents included here have already been obtained by J.M. Aguirregabiria [9] for the exterior case and by P. Teyssandier [10] for both the exterior and interior cases. Those results, as well as some new other ones, are introduced to make a comparison with the harmonic scenario and because until now they have not been published in their entirety.

The paper is completed with an Appendix which is devoted to explaining the Heun differential equation; this has been included because of the rather odd and at the same time fundamental equation involved. We hope that with this the reader will find it easy to understand Section 3.

2. - Complete model of the Schwarzschild metric

The interior Schwarzschild metric has the following expression in standard polar coordinates $\{t, r, \theta, \varphi\}$

$$ds_I^2 = - \left[\frac{3}{2} \gamma^{1/2}(r_0) - \frac{1}{2} \gamma^{1/2}(r) \right]^2 dt^2 + \gamma^{-1}(r) dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (1)$$

where the radial coordinate r is assumed to be restricted to the interval $[0, r_0]$, r_0 being the radius of the star. The following notation is used:

$$\gamma(r) \equiv 1 - \frac{r^2}{L^2} \quad , \quad \frac{1}{L^2} \equiv \frac{1}{3} \chi \rho \quad (r_0 < L) \quad , \quad \chi \equiv 8\pi \quad (2)$$

ρ being the energy density (constant) of the star and where we are dealing with geometrized units ($G = c = 1$). As is known, the pressure of the model is given as a function of the radius by the following expression

$$p(r) = \rho \frac{\gamma^{\frac{1}{2}}(r) - \gamma^{\frac{1}{2}}(r_0)}{3\gamma^{\frac{1}{2}}(r_0) - \gamma^{\frac{1}{2}}(r)} \quad (3)$$

which provides the following values for the pressure at the surface and the center of the star respectively

$$\begin{cases} p(r_0) = 0 \\ p(0) \equiv p_c = \rho \frac{1 - \gamma^{\frac{1}{2}}(r_0)}{3\gamma^{\frac{1}{2}}(r_0) - 1} \end{cases} \quad (4)$$

Since the value p_c of the pressure at the center of the star must be finite, it relates the radius of the star r_0 and the density parameter L by means of the following restriction

$$3\gamma^{\frac{1}{2}}(r_0) - 1 > 0 \quad \Rightarrow \quad \frac{r_0}{L} < \frac{2\sqrt{2}}{3} \quad (5)$$

The exterior Schwarzschild metric is written in the same set of standard polar coordinates as follows ($r \geq r_0$)

$$ds_E^2 = - \left(1 - \frac{2m}{r} \right) dt^2 + \left(1 - \frac{2m}{r} \right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (6)$$

where m represents the mass of the star, which is related to the density ρ and the radius r_0 by the known expression

$$m \equiv \frac{4}{3} \pi r_0^3 \rho \quad \left(\Leftrightarrow \frac{2m}{r_0} = \frac{r_0^2}{L^2} < \frac{8}{9} \right) \quad (7)$$

The above expressions for the Schwarzschild metric (interior and exterior) are such that the Darmois matching conditions [6] are automatically fulfilled on the surface of the star ($\Sigma : r = r_0$). These conditions –that is, the continuity of both the first and the second fundamental forms– will be symbolically denoted as follows

$$\begin{cases} \text{I} : [h_{ab}]_{\Sigma} = 0 \\ \text{II} : [K_{ab}]_{\Sigma} = 0 \end{cases}, \quad (a, b, \dots = 0, 2, 3 = t, \theta, \varphi) \quad (8)$$

In what follows we make explicit the expression of the fundamental forms to show the requirements provided by (8).

- *The first fundamental form is*

$$h_{ab}(p) \equiv g_{\alpha\beta}[x(p)] e_{a/}^{\alpha} e_{b/}^{\beta} \quad (9)$$

$e_{a/}^{\alpha}$ being the tangent vectors to the surface Σ

$$e_{a/}^{\alpha}(p^b) \equiv \frac{\partial x^{\alpha}}{\partial p^b} \quad (\Sigma : x^0 = p^0 = t, x^2 = p^2 = \theta, x^3 = p^3 = \varphi) \quad (10a)$$

which leads to

$$(h_{ab}) = \begin{pmatrix} g_{tt} & 0 & 0 \\ 0 & g_{\theta\theta} & 0 \\ 0 & 0 & g_{\varphi\varphi} \end{pmatrix} \quad (10b)$$

i.e., g_{tt} must be continuous, g_{rr} being unrestricted (note that the coordinates satisfy the Euclidean sphere condition everywhere: $g_{\varphi\varphi} = g_{\theta\theta} \sin^2 \theta = r^2 \sin^2 \theta$). We shall see below that the continuity of g_{rr} is imposed by the continuity of the second fundamental form.

- *The second fundamental form is*

$$K_{ab} \equiv -l_{\mu} e_{a/}^{\rho} \nabla_{\rho} e_{b/}^{\mu} = -l_{\alpha} \left(\frac{\partial e_{a/}^{\alpha}}{\partial p^b} + \Gamma_{\lambda\mu}^{\alpha} e_{a/}^{\lambda} e_{b/}^{\mu} \right) = -\Gamma_{ab}^r \quad (11)$$

where the notation $l_{\alpha} \equiv \partial_{\alpha}(r - r_0)$ has been used to denote the normal vector to the surface Σ

$$(K_{ab}) = \frac{1}{2} g^{rr} \begin{pmatrix} \partial_r g_{tt} & 0 & 0 \\ 0 & \partial_r g_{\theta\theta} & 0 \\ 0 & 0 & \partial_r g_{\varphi\varphi} \end{pmatrix} \quad (12)$$

It may be deduced from (12) that both g_{rr} and $\partial_r g_{tt}$ must be continuous. As a summary, only the discontinuity of $\partial_r g_{rr}$ is allowed, which following (1) and (6) turns out to be

$$[\partial_r g_{rr}]_\Sigma \equiv \partial_r g_{rr}^E(r_0) - \partial_r g_{rr}^I(r_0) = -3 \frac{r_0}{L^2} \gamma^{-2}(r_0) \quad (13)$$

It is then obvious that the set of standard polar coordinates $\{t, r, \theta, \varphi\}$ describing the complete Schwarzschild model are not admissible in the sense of Lichnerowicz, since the metric is not C^1 .

3. - Global asymptotically Cartesian harmonic coordinates

The aim of this section is to find a global system of harmonic coordinates for the complete Schwarzschild model, which must be asymptotically Cartesian and such that the components of the metric are C^1 at the boundary $\Sigma : r = r_0$. Let us consider the generic form of the Schwarzschild metric in standard coordinates (1),(6)

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (14)$$

We now perform a change of coordinates

$$\{t, r, \theta, \varphi\} \equiv \{x^\alpha\} \longrightarrow \{\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}\} \equiv \{x^{\tilde{\alpha}}\} \quad (15)$$

satisfying the following conditions

$$\begin{cases} \partial_\lambda (\sqrt{-g} g^{\lambda\mu} \partial_\mu x^{\tilde{\alpha}}) = 0 \\ \lim_{\tilde{r} \rightarrow \infty} g_{\tilde{\alpha}\tilde{\beta}} = \eta_{\alpha\beta} \equiv \text{diag}(-1, +1, +1, +1) \end{cases} \quad (16)$$

which represent the harmonicity condition and asymptotic cartesian behaviour respectively, and where $\tilde{r} \equiv \sqrt{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2}$.

The easiest way to solve the problem is by using polar coordinates $\{t, \tilde{r}, \tilde{\theta}, \tilde{\varphi}\}$ associated with the harmonic ones $\{x^{\tilde{\alpha}}\}$

$$\begin{cases} \tilde{x} + i \tilde{y} = \tilde{r} \sin \tilde{\theta} e^{i\tilde{\varphi}} \\ \tilde{z} = \tilde{r} \cos \tilde{\theta} \end{cases} \quad (17)$$

We see that in order to preserve the diagonal structure as well as the structure of the spherical symmetry, the change of coordinates (15) should be as follows

$$\begin{aligned} \tilde{t} = t \quad , \quad \tilde{r} = f(r) \quad , \quad \tilde{\theta} = \theta \quad , \quad \tilde{\varphi} = \varphi \\ \left(\lim_{r \rightarrow \infty} \frac{f(r)}{r} = 1 \right) \end{aligned} \quad (18)$$

Henceforth, equation (16) turns out to be the following second-order linear differential equation for the function $f(r)$:

$$\frac{d}{dr} \left(r^2 \sqrt{\frac{-g_{tt}}{g_{rr}}} \frac{df}{dr} \right) - 2\sqrt{-g_{tt} g_{rr}} f = 0 \quad (19)$$

Let us now see the form of this equation in the interior and at the exterior of the star.

A) The exterior problem

For the expressions of the exterior Schwarzschild metric given (6), equation (19) is written as follows

$$r(r - 2m)f''(r) + 2(r - m)f'(r) - 2f(r) = 0 \quad (20)$$

whose general solution is as follows [4]

$$\tilde{r} = f_{\text{ext}}(r) = Q_1(r - m) + Q_{\text{ext}} \tilde{g}(r) \quad (21)$$

where Q_1 y Q_{ext} are arbitrary constants and the function $\tilde{g}(r)$ is defined as

$$\tilde{g}(r) \equiv (r - m) \log \left(1 - \frac{2m}{r} \right) + 2m \quad (22)$$

Looking at the asymptotic condition, we have

$$f_{\text{ext}}(r \rightarrow \infty) \sim Q_1 r \left(1 - \frac{m}{r} \right) + Q_{\text{ext}} \frac{2m^2}{r} \sim Q_1 r \quad (23)$$

and hence the constant Q_1 must be equal to one ($Q_1 = 1$).

B) The interior problem

By now introducing expression (1) from the interior metric into the generic equation (19), we obtain the following differential equation for the function $f(r)$

$$r^2 \gamma(r) f''(r) + r \left[2\gamma(r) - \frac{r^2}{L^2} \frac{3\gamma^{\frac{1}{2}}(r_0) - 2\gamma^{\frac{1}{2}}(r)}{3\gamma^{\frac{1}{2}}(r_0) - \gamma^{\frac{1}{2}}(r)} \right] f'(r) - 2f(r) = 0 \quad (24)$$

which is at first sight difficult to tackle. Nevertheless, the change of the independent variable

$$r \longrightarrow x = 1 - \gamma^{\frac{1}{2}}(r) : \begin{cases} r \in [0, r_0] \\ x \in [0, x_0 \equiv 1 - \gamma^{\frac{1}{2}}(r_0)] \end{cases} \quad (25)$$

leads to the following equation

$$f''(x) + \frac{4x^2 - (5 + 3b)x + 3b}{(x - b)x(x - 2)} f'(x) - \frac{2}{x^2(x - 2)^2} f(x) = 0 \quad (26)$$

where primes now denote derivatives with respect to the new variable x and where the following definition has been used

$$b \equiv 1 - 3\gamma^{\frac{1}{2}}(r_0) \quad (27)$$

Let us note here what restrictions the upper limit of the parameter μ_0 implies:

$$0 < \mu_0 < \frac{8}{9} \quad \Rightarrow \quad \begin{cases} 0 < x_0 < \frac{2}{3} \\ -2 < b < 0 \end{cases} \quad (28)$$

This new differential equation (26) has four singular points, all of which are regular

$$\{x = b, x = 0, x = 2, x = \infty\} \quad (29)$$

Now, since we are interested in solutions with good behaviour in the $[0, x_0]$ interval, we look for analytical solutions at the origin as a Frobenius series. This procedure leads to a single analytical series whose exponent is $E_f = 1/2$. However, this must be constructed by means of a rather awkward recursive expression. Accordingly, we have chosen to reduce equation (26) into the canonical Heun form [5] (see Appendix), which allows us to obtain the required solution by using some properties of that equation. The final result for the general and analytical solution at the origin is the following convergent expression in the $[0, x_0]$ interval

$$f_{\text{int}}(r) = Q_{\text{int}} r H\left(\frac{2}{b}, 3 + \frac{1}{b}; 4, 1, \frac{5}{2}, 1; \frac{x}{b}\right) \quad (30)$$

Q_{int} being an arbitrary constant and where $H(a, q; \alpha, \beta, \gamma, \delta; z)$ is the so-called Heun series, defined as follows

$$H(a, q; \alpha, \beta, \gamma, \delta; z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (31)$$

with the recursive law ($n = 0, 1, 2, \dots$)

$$\begin{aligned} & a(n+2)(\gamma + n + 1)c_{n+2} \\ & - (n+1) \left[\alpha + \beta - \delta + n + 1 + a(\gamma + \delta + n) + \frac{q}{n+1} \right] c_{n+1} \\ & + [n(\alpha + \beta + n) + \alpha\beta] c_n = 0 \end{aligned}$$

$$c_0 = 1 \quad , \quad c_1 = \frac{q}{a\gamma} \quad (32)$$

which is convergent iff

$$|a| \geq 1 \quad , \quad |z| < 1 \quad (33)$$

Note that in our case we have

$$1 < \left| \frac{2}{b} \right| < \infty \quad , \quad \left| \frac{x}{b} \right| < 1 \quad \text{if} \quad \mu_0 < \frac{3}{4} \quad (34)$$

i.e., the series is convergent everywhere inside the star only if its radius is larger than the Schwarzschild radius by a factor $4/3$. For example, a typical neutron star of 1.4 solar masses and a radius of $2 \times 10^4 m$ gives a parameter

$$\mu_0 = \frac{2m}{r_0} = \frac{1.4 \times 1.5}{20} = 0.15 < 0.75 \quad (35)$$

Thus, $\mu_0 < 3/4$ is a high upper limit that does not imply restrictions to a realistic star model and hence solution (30) becomes broadly general.

By writing solution (30) as a power series in the parameter r/L , we find the following expression

$$\begin{aligned} f_{\text{int}}(r) = Q_{\text{int}} r \left(1 + \frac{1+3b}{10b} \frac{r^2}{L^2} + \frac{8+19b+45b^2}{280b^2} \frac{r^4}{L^4} \right. \\ \left. + \frac{152+424b+729b^2+1575b^3}{15120b^3} \frac{r^6}{L^6} + \dots \right) \end{aligned} \quad (36)$$

• *Quan's case*

$$\mu_0 \equiv \frac{r_0^2}{L^2} = \frac{2m}{r_0} = \frac{8}{9} \quad (\Leftrightarrow b = 0) \quad (37)$$

For this case, solution (30),(36) is strongly divergent and we must return to the original differential equation (24), taking the parameter b to be zero ($3\gamma^{\frac{1}{2}}(r_0) = 1$). The resulting equation only has three regular singular points (one of them at infinity) and hence reduces to a hypergeometric equation by using the standard procedure (see [4] for the final result).

C) Admissible Lichnerowicz coordinates

We shall prove that it is possible to set the constants Q_{int} and Q_{ext} in such a way that the components of the metric and their derivatives, written in the new set of coordinates, are continuous on the surface of the star ($r = r_0$). In order to do so, we only require the function $f(r)$ defining the change of coordinates to be C^1 at that boundary, i.e.,

$$[f]_{\Sigma} = [f']_{\Sigma} = 0 \quad \Leftrightarrow \quad \begin{cases} f_{\text{ext}}(r_0) = f_{\text{int}}(r_0) \\ f'_{\text{ext}}(r_0) = f'_{\text{int}}(r_0) \end{cases} \quad (38)$$

which, in agreement with (21-22), (30), can be translated into the following relations

$$\begin{aligned} r_0 - m + \tilde{g}(r_0) Q_{\text{ext}} &= r_0 \tilde{h}(r_0) Q_{\text{int}} \\ 1 + \tilde{g}'(r_0) Q_{\text{ext}} &= [\tilde{h}(r_0) + r_0 \tilde{h}'(r_0)] Q_{\text{int}} \end{aligned} \quad (39)$$

with

$$\tilde{h}(r) \equiv H\left(\frac{2}{b}, 3 + \frac{1}{b}; 4, 1, \frac{5}{2}, 1; \frac{x}{b}\right) \quad (40)$$

The set of equations (39) constitutes an algebraic linear system for the unknown variables Q_{int} and Q_{ext} , whose solution is

$$\begin{cases} Q_{\text{ext}} = \frac{(r_0 - m)[\tilde{h}(r_0) + r_0 \tilde{h}'(r_0)] - r_0 \tilde{h}(r_0)}{r_0 \tilde{h}(r_0) \tilde{g}'(r_0) - [\tilde{h}(r_0) + r_0 \tilde{h}'(r_0)] \tilde{g}(r_0)} \\ Q_{\text{int}} = \frac{(r_0 - m) \tilde{g}'(r_0) - \tilde{g}(r_0)}{r_0 \tilde{h}(r_0) \tilde{g}'(r_0) - [\tilde{h}(r_0) + r_0 \tilde{h}'(r_0)] \tilde{g}(r_0)} \end{cases} \quad (41)$$

which is well defined, since the denominator of both expressions is trivially different from zero. We now give the firsts terms of the expansion in power series of the μ_0 parameter for the values of those constants

$$\begin{cases} 2mQ_{\text{ext}} = r_0 \left(\frac{12}{35} - \frac{4}{21}\mu_0 - \frac{58}{1155}\mu_0^2 - \frac{136}{5005}\mu_0^3 + \dots \right) \\ Q_{\text{int}} = 1 - \frac{3}{4}\mu_0 - \frac{1}{16}\mu_0^2 - \frac{1}{96}\mu_0^3 + \frac{1439}{134400}\mu_0^4 \dots \end{cases} \quad (42)$$

It is worth noticing that these values have nothing to do with those obtained by Quan for the case $\mu_0 = 8/9$, since our results require an upper limit for the parameter $\mu_0 < 3/4$, and hence no comparison can be made.

Let us now check that this choice of constants does indeed lead to a C^1 metric, written in harmonic polar coordinates, at the boundary $r = r_0$. By making use of the change (15-18), we have

$$ds^2 = g_{tt}(r)dt^2 + \frac{g_{rr}(r)}{f'^2(r)}d\tilde{r}^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad , \quad [r = r(\tilde{r})] \quad (43)$$

Obviously, all the components of the metric as well as the derivatives of g_{tt} , $g_{\theta\theta}$ and $g_{\varphi\varphi}$ are continuous. Regarding the derivative of $g_{\tilde{r}\tilde{r}}$, this turns out to be

$$[\partial_{\tilde{r}} g_{\tilde{r}\tilde{r}}]_{\Sigma} = \frac{1}{f'} \left[\left(\frac{g_{rr}}{f'^2} \right)' \right]_{\Sigma} = \frac{1}{f'^4} \left([\partial_r g_{rr}]_{\Sigma} f'(r_0) - 2g_{rr}(r_0) [f'']_{\Sigma} \right) \quad (44)$$

However, taking into account the differential equations (20) and (24) for the exterior and the interior respectively, we have that

$$[f'']_{\Sigma} = -\frac{3}{2} \frac{r_0}{L^2} \gamma^{-1}(r_0) f'(r_0) \quad (45)$$

and therefore, since the discontinuity of $\partial_r g_{rr}$ is given by (13), we finally have that

$$[\partial_{\tilde{r}} g_{\tilde{r}\tilde{r}}]_{\Sigma} = 0 \quad (46)$$

which is what we wished to prove. Therefore, the harmonic coordinates obtained with the constants (41-42) are admissible coordinates in the sense of Lichnerowicz.

4. - Interpretation of Quan's exterior constant

A) Multipole expansion

As already mentioned in the Introduction, the purpose of this Section is to search for the physical meaning of the constant Q_{ext} associated with the choice of global harmonic coordinates given in the previous Section. We first carry out a multipole expansion of the exterior Schwarzschild metric in those coordinates, i.e., a power series in the inverse of the radial coordinate \tilde{r} . This expansion can be taken, up to order five, from a previous paper [7], where it was introduced for other purposes. Here we write the metric on a slightly different basis that is useful for the following calculations.

$$ds^2 = T(\tilde{r}) dt^2 + \left[A(\tilde{r}) \delta_{ij} + B(\tilde{r}) n_i n_j \right] d\tilde{x}^i d\tilde{x}^j \quad (47)$$

with

$$n^i \equiv \frac{\tilde{x}^i}{\tilde{r}} \quad (n_i \equiv \delta_{ij} n^j) \quad (48)$$

$$T \equiv g_{tt}(r) \quad , \quad A \equiv \frac{r^2}{\tilde{r}^2} \quad , \quad B \equiv \frac{g_{rr}(r)}{f'^2(r)} - \frac{r^2}{\tilde{r}^2} \quad \left[r = r(\tilde{r}) \right] \quad (49)$$

By using these definitions and the results from [7], we obtain the following expansion up to the order $1/\tilde{r}^5$

$$\begin{aligned} T &= -1 + 2\frac{m}{\tilde{r}} - 2\frac{m^2}{\tilde{r}^2} + 2\frac{m^3}{\tilde{r}^3} - 2\frac{Km + m^4}{\tilde{r}^4} \\ &\quad + 2\frac{2Km^2 + m^5}{\tilde{r}^5} + \dots \\ A &= 1 + 2\frac{m}{\tilde{r}} + \frac{m^2}{\tilde{r}^2} + 2\frac{K}{\tilde{r}^3} + 2\frac{Km}{\tilde{r}^4} + \frac{6}{5}\frac{Km^2}{\tilde{r}^5} + \dots \\ B &= \frac{m^2}{\tilde{r}^2} + 2\frac{-3K + m^3}{\tilde{r}^3} + 2\frac{-6Km + m^4}{\tilde{r}^4} \\ &\quad + 2\frac{-9Km^2 + m^5}{\tilde{r}^5} + \dots \end{aligned} \quad (50)$$

where a new constant K is introduced, which is related with Q_{ext} as follows

$$K \equiv \frac{2}{3} m^3 Q_{\text{ext}} \quad (51)$$

B) Linear approximation in harmonic coordinates

As is well known [11], the Einstein equations can be written in the following way

$$\partial_{\lambda\mu}(\mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\lambda\mu} - \mathfrak{g}^{\alpha\lambda}\mathfrak{g}^{\beta\mu}) = 16\pi G(-g) \left(T^{\alpha\beta} + t_L^{\alpha\beta} \right) \quad (52)$$

$\mathfrak{g}^{\alpha\beta} \equiv \sqrt{-g} g^{\alpha\beta}$ being the contravariant metric density, $g \equiv \det(g_{\alpha\beta})$ the determinant of the metric, and $t_L^{\alpha\beta}$ the Landau–Lifshitz energy-momentum pseudotensor. One of the procedures used in the past to solve these equations is the so-called perturbative postminkowskian algorithm [12]; starting from a system of asymptotically Cartesian coordinates, this method consists in looking for a solution as a formal power series in the gravitational constant G :

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + \sum_{n=1}^{\infty} G^n \mathfrak{h}^{\alpha\beta}_n \quad (53)$$

and taking for the coordinates the harmonicity condition

$$\partial_\alpha \mathfrak{g}^{\alpha\beta} = 0 \quad (54)$$

In particular, this procedure has been of great interest for the study of certain aspects of gravitational radiation [12]. We are now dealing with a vacuum stationary scenario—that is, $\mathfrak{h}^{\alpha\beta}$ is independent of time—and hence the equations resulting in the linear approximation are as follows

$$\begin{cases} \Delta \mathfrak{h}^{\lambda\mu}_1 = 0 \\ \partial_i \mathfrak{h}^{i\mu}_1 = 0 \end{cases} \quad (55)$$

whose general solution can be written in the following way (see, for instance, [13])

$$\mathfrak{h}^{\lambda\mu}_1 = \mathfrak{h}^{\lambda\mu}_{\text{can}} + \partial^\lambda w^\mu + \partial^\mu w^\lambda - \eta^{\lambda\mu} \partial_\rho w^\rho \quad , \quad (\Delta w_\lambda = 0) \quad (56)$$

where the canonical part $\mathfrak{h}^{\lambda\mu}_{\text{can}}$ is given by

$$\begin{cases} \mathfrak{h}^{00}_{\text{can}} = -4 \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \sum_{m=-l}^{+l} M_l^m Y_l^m(\theta, \varphi) \\ \mathfrak{h}^{0j}_{\text{can}} = +4 \sum_{l=1}^{\infty} \frac{1}{r^{l+1}} \sum_{m=-l}^{+l} S_l^m Y_{l,l}^{m,j}(\theta, \varphi) \\ \mathfrak{h}^{jk}_{\text{can}} = 0 \end{cases} \quad (57)$$

M_l^m and S_l^m being the Geroch–Hansen static and dynamical multipole moments [14] (up to a numerical constant factor), and where the usual notation has been used for spherical harmonics. Regarding the “gauge” part of the solution [terms involving the function w^ρ in (56)], we have the following expressions

$$w^0 = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \sum_{m=-l}^{+l} C_l^m Y_l^m(\theta, \varphi) \quad (58a)$$

$$\begin{aligned} w^k = & \sum_{l=1}^{\infty} \frac{1}{r^l} \sum_{m=-l}^{+l} H_l^{m,k} Y_l^m(\theta, \varphi) + \sum_{l=1}^{\infty} \frac{1}{r^{l+1}} \sum_{m=-l}^{+l} J_l^m Y_{l,l}^{m,k}(\theta, \varphi) \\ & - \sum_{l=0}^{\infty} \frac{l+1}{r^{l+2}} \sum_{m=-l}^{+l} K_l^m Y_l^m(\theta, \varphi) n^k + \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \sum_{m=-l}^{+l} K_l^m \partial^k Y_l^m(\theta, \varphi) \end{aligned} \quad (58b)$$

where C_l^m , $H_l^{m,k}$, J_l^m y K_l^m are new constants (scalar or vectorial) whose meanings have not yet been discussed.

The general solution of equations (55) with spherical symmetry is obtained from the previous expressions (57-58) by restricting ourselves to monopolar terms, i.e, by taking into account only $l = 0$ in the above series:

$$\left\{ \begin{array}{l} {}^1\mathfrak{h}_{\text{can}}^{00} = -4\frac{m}{r} \\ {}^1\mathfrak{h}_{\text{can}}^{0j} = 0 \\ {}^1\mathfrak{h}_{\text{can}}^{ij} = 0 \end{array} \right. \quad (m \equiv M_0^0) \quad , \quad \left\{ \begin{array}{l} w^0 = \frac{C}{r} \\ w^j = -\frac{K}{r^2} n^j \end{array} \right. \quad (C \equiv C_0^0) \quad (K \equiv K_0^0) \quad (59)$$

In agreement with (56), this expressions lead to the following solution

$$\left\{ \begin{array}{l} {}^1\mathfrak{h}^{00} = -4\frac{m}{r} \\ {}^1\mathfrak{h}^{0j} = \partial^j w^0 \\ {}^1\mathfrak{h}^{ij} = -2\frac{K}{r^3} \delta^{ij} + 6\frac{K}{r^3} n^i n^j \end{array} \right. \quad (60)$$

Finally, the linear order of the metric is obtained by using the known relation

$$g_{\lambda\mu} = -{}^1\mathfrak{h}_{\lambda\mu} + \frac{1}{2} {}^1\mathfrak{h} \eta_{\lambda\mu} \quad \left({}^1\mathfrak{h} \equiv \eta_{\lambda\mu} {}^1\mathfrak{h}^{\lambda\mu} \right) \quad (61)$$

which explicitly turns out to be

$$\begin{cases} g_{00}^1 = 2\frac{m}{r} \\ g_{0j}^1 = \partial_j w^0 = 0 \quad , \quad (C = 0) \\ g_{jk}^1 = \left(2\frac{m}{r} + 2\frac{K}{r^3}\right) \delta_{jk} - 6\frac{K}{r^3} n_j n_k \end{cases} \quad (62)$$

where we have taken $C = 0$ since g_{0j}^1 is a gradient and can therefore be omitted (static condition).

Let us notice that, as it should, metric (62) contains all the linear terms of the Schwarzschild metric written in harmonic coordinates (47),(50) shown in the previous Section. This detail, which might appear insignificant, turns out to be the relevant point for understanding the meaning of the constant K , as we shall now see.

C) A spherically symmetric “singular” source

A significant feature of the metric (62) is that it proves to be a solution of the linearized Einstein equations with the following singular energy- momentum tensor (order zero) on the right hand side of the equations,

$$\begin{cases} T^{00}{}^0 = m \delta(\vec{x}) \\ T^{0j}{}^0 = 0 \\ T^{ij}{}^0 = \frac{1}{2}K \left[\delta^{ij} \delta^{kl} - \delta^{i(k} \delta^{l)j} \right] \partial_{kl} \delta(\vec{x}) \end{cases} \quad (63)$$

Thus, the equations can be written as follows:

$$\begin{cases} \Delta \mathfrak{h}^{00} = 16\pi m \delta(\vec{x}) \\ \Delta \mathfrak{h}^{ij} = 8\pi K \left[\delta^{ij} \Delta \delta(\vec{x}) - \partial^{ij} \delta(\vec{x}) \right] \end{cases} \quad (64)$$

and we can solve them by using the Poisson integrals to give

$$\begin{cases} \mathfrak{h}^{00}(\vec{x}) = -4m \int \frac{\delta(\vec{y})}{|\vec{x} - \vec{y}|} d^3 \vec{y} = -4\frac{m}{r} \\ \mathfrak{h}^{ij}(\vec{x}) = +2K \int \frac{\partial^{ij} \delta(\vec{y})}{|\vec{x} - \vec{y}|} d^3 \vec{y} = -2\frac{K}{r^3} \delta^{ij} + 6\frac{K}{r^3} n^i n^j \end{cases} \quad (65)$$

which is merely the metric density obtained before (60).

Let us now calculate the *stress quadrupolar moment* of this energy- momentum tensor (63)

$$\begin{aligned} \int x^p x^q T^{ij} d^3 \vec{x} &= \frac{1}{2} K \left[\delta^{ij} \delta^{kl} - \delta^{i(k} \delta^{l)j} \right] \int x^p x^q \partial_{kl} \delta(\vec{x}) d^3 \vec{x} \\ &= K \left[\delta^{ij} \delta^{pq} - \delta^{i(p} \delta^{q)j} \right] \end{aligned} \quad (66)$$

We see that the constant K , and therefore Q_{ext} , are related to the stress quadrupolar moment of the source. In our opinion, this is simply a first argument to understand the role played by this kind of constant, which are not present when the Thorne gauge is used for the analysis of standard multipole moments. In the future we hope to develop a coherent theory able to completely justify the interpretation made.

5. - Global asymptotically Cartesian q-harmonic coordinates

In Section 3 we looked for a global system of harmonic coordinates that would be asymptotically Cartesian and admissible in the sense on Lichnerowicz. This issue can also be solved for a system of the so-called q-harmonic coordinates. These coordinates were introduced by Bel [8] to analyze reference frames in General Relativity as congruences of time-like curves. Although the following results are (almost all) from other authors, we shall show them briefly because they have not been published by those authors.

Q-harmonic coordinates are only meaningful when associated with the also so-called q-harmonic congruences [8]. Since the Schwarzschild metric admits as a q-harmonic congruence the Killing time congruence, we restrict ourselves to recalling the definition of q-harmonic coordinates for this case. The change of coordinates from the standard ones would be as follows

$$\{r, \theta, \varphi\} \equiv \{x^i\} \longrightarrow \{\bar{x}, \bar{y}, \bar{z}\} \equiv \{x^{\bar{i}}\} \quad (67)$$

with the q-harmonicity condition

$$\begin{cases} \partial_i \left(\sqrt{\hat{g}} g^{ij} \partial_j x^{\bar{k}} \right) = 0 \\ \lim_{\bar{r} \rightarrow \infty} g_{\bar{i} \bar{j}} = \delta_{ij} \end{cases} \quad (68)$$

where $\bar{r} \equiv \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}$, and \hat{g} represents the determinant of the spatial metric of the quotient space defined by the Killing congruence.

As in the harmonic case, associated polar coordinates are defined by

$$\begin{cases} \bar{x} + i\bar{y} = \bar{r} \sin \bar{\theta} e^{i\bar{\varphi}} \\ \bar{z} = \bar{r} \cos \bar{\theta} \end{cases} \quad (69)$$

$$\bar{r} = f(r) \quad , \quad \bar{\theta} = \theta \quad , \quad \bar{\varphi} = \varphi$$

$$\left(\lim_{r \rightarrow \infty} \frac{f(r)}{r} = 1 \right)$$

and this leads to the following differential equation

$$\frac{d}{dr} \left(\frac{r^2}{\sqrt{g_{rr}}} \frac{df}{dr} \right) - 2\sqrt{g_{rr}} f = 0 \quad (71)$$

A) The exterior problem (q-harmonic coordinates)

In this case the previous differential equation (71) is written as follows

$$r(r - 2m)f''(r) + 2\left(r - \frac{3}{2}m\right)f'(r) - 2f(r) = 0 \quad (72)$$

whose general solution was already obtained by Aguirregabiria [9]

$$\bar{r} = f_{\text{ext}}(r) = J_1 \left(r - \frac{3}{2}m \right) + J_2 \bar{g}(r) \quad (73)$$

where J_1 and J_2 are constants, and the function $\bar{g}(r)$ is defined by

$$\bar{g}(r) \equiv \left(r - \frac{1}{2}m \right) \sqrt{1 - \frac{2m}{r}} \quad (74)$$

with the following asymptotic behavior

$$\bar{g}(r) = r \left(1 - \frac{3}{2} \frac{m}{r} - \frac{1}{4} \frac{m^3}{r^3} + \dots \right) \quad (75)$$

Thus, according to (73) we have

$$J_1 + J_2 = 1 \quad (76)$$

B) The interior problem (q-harmonic coordinates)

For this case, the differential equation (71) has the following form

$$r^2 \left(1 - \frac{r^2}{L^2} \right) f''(r) + r \left(2 - \frac{3r^2}{L^2} \right) f'(r) - 2f(r) = 0 \quad (77)$$

whose unique analytical solution at the origin has already been obtained by Teyssandier [10]

$$f_{\text{int}}(r) = P \frac{r}{L} F \left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}; \frac{r^2}{L^2} \right)$$

$$= P \frac{r}{L} \left(1 + \frac{3}{10} \frac{r^2}{L^2} + \frac{9}{56} \frac{r^4}{L^4} + \frac{5}{48} \frac{r^6}{L^6} + \dots \right) \quad (78)$$

P being an arbitrary constant, and where $F(a, b, c, ; x)$ represents the usual hypergeometric function whose first terms of its power expansion are shown for clarity of expression.

C) Continuity at the boundary Σ

As in the harmonic case, the arbitrary constants P and J_2 (or J_1) can be fixed by imposing that the function $f(r)$ be C^1 on the surface of the star, i.e.,

$$[f]_{\Sigma} = [f']_{\Sigma} = 0 : \begin{cases} \left(r_0 - \frac{3}{2}m\right) J_1 + \bar{g}(r_0) J_2 = r_0 \bar{h}(r_0) \frac{P}{L} \\ J_1 + \bar{g}'(r_0) J_2 = [\bar{h}(r_0) + r_0 \bar{h}'(r_0)] \frac{P}{L} \end{cases} \quad (79)$$

with

$$\bar{h}(r) \equiv F\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{r^2}{L^2}\right) \quad (80)$$

These conditions lead to the following expressions for the constants

$$\begin{cases} J_2 = \frac{(r_0 - 3m/2) [\bar{h}(r_0) + r_0 \bar{h}'(r_0)] - r_0 \bar{h}(r_0)}{r_0 \bar{h}(r_0) [\bar{g}'(r_0) - 1] - [\bar{h}(r_0) + r_0 \bar{h}'(r_0)] [\bar{g}(r_0) - (r_0 - 3m/2)]} \\ \frac{P}{L} = \frac{(r_0 - 3m/2) \bar{g}'(r_0) - \bar{g}(r_0)}{r_0 \bar{h}(r_0) [\bar{g}'(r_0) - 1] - [\bar{h}(r_0) + r_0 \bar{h}'(r_0)] [\bar{g}(r_0) - (r_0 - 3m/2)]} \end{cases} \quad (81)$$

whose expansion in power series of the parameter $\mu_0 \equiv 2m/r_0 = r_0^2/L^2$ is given by the following expressions

$$\begin{cases} 4m^2 J_2 = r_0^2 \left[\frac{-8}{5} + \frac{72}{35} \mu_0 - \frac{29}{1050} \mu_0^2 - \frac{8}{825} \mu_0^3 + O(\mu_0^4) \right] \\ \frac{P}{L} = 1 - \mu_0 + \frac{9}{80} \mu_0^2 + \frac{1}{280} \mu_0^3 + O(\mu_0^4) \end{cases} \quad (82)$$

Finally, it should be mentioned that Teyssandier [10] has proved, in a similar way to that used here with the harmonic coordinates, that these values of the constants provide a C^1 metric on the surface Σ , i.e., the q-harmonic coordinates obtained are also admissible coordinates in the sense of Lichnerowicz.

6. - Conclusions

We have presented a global solution to Einstein equations for static spherically symmetric perfect fluid with homogeneous energy density distribution. At the outside of matter distribution we have, as it should be, the Schwarzschild space-time, whereas inside, our solution coincides with the well-known Schwarzschild solution. The novelty

of our approach being that the solution is written globally in harmonic and q-harmonic coordinates. At the boundary surface the Lichnerowicz junction conditions are satisfied, which implies that, both, the metric and its first derivatives are continuous across the boundary. This, in turn, implies that the exterior space-time contains an additional parameter, besides the mass. This parameter being related to the stress quadrupolar moment [see (66)]. Obviously this new parameter may be eliminated by means of a coordinate transformation. All the above brings out two facts: on one hand, the Schwarzschild space-time may be produced by a variety of sources (which is well-known) and on the other hand, when the solution is written globally, using Lichnerowicz junction conditions, it contains an additional parameter reflecting properties of the source, other than the mass.

Appendix :

Heun's equation

A) The so-called Heun equation is the following second-order linear differential equation

$$x(x-1)(x-a)F''(x) + \{(\alpha + \beta + 1)x^2 - [\alpha + \beta + 1 + a(\gamma + \delta) - \delta]x + a\gamma\}F'(x) + (\alpha\beta x - q)F(x) = 0 \quad (A1)$$

where $(a, q; \alpha, \beta, \gamma, \delta)$ are numerical constants. This equation has the following singular points, all of which are regular

$$\{x = 0, x = 1, x = a, x = \infty\} \quad (A2)$$

We restrict ourselves to the point $x = 0$, which is the one of interest in the main text, and one readily sees that the Frobenius series type solutions have exponents $(0, 1 - \gamma)$.

For the zero exponent, the solution is

$$F(x) = H(a, q; \alpha, \beta, \gamma, \delta; x) \quad , \quad (\gamma \neq 0, -1, -2, \dots) \quad (A3)$$

H being the Heun series (31) appearing in the main text, which is convergent if $|a| \geq 1$ and with a radius of convergence $|x| < 1$.

For the exponent $1 - \gamma$, the solution is

$$F(x) = x^{1-\gamma} H(a, q_1; \alpha_1, \beta_1, \gamma_1, \delta; x) \quad , \quad (\gamma \neq 1, 2, 3, \dots) \quad (A4)$$

with

$$\begin{cases} q_1 \equiv q + (1 - \gamma)(\alpha + \beta + 1 - \gamma - \delta + a\delta) \\ \alpha_1 \equiv \alpha + 1 - \gamma \quad , \quad \beta_1 \equiv \beta + 1 - \gamma \quad , \quad \gamma_1 \equiv 2 - \gamma \end{cases} \quad (A5)$$

When the previous series are not convergent, i.e., $|a| < 1$, we perform the change of variable

$$x \longrightarrow \bar{x} = \bar{a}x \quad , \quad \left(\bar{a} \equiv \frac{1}{a} \right) \quad (A6)$$

and this leads to the following solution

$$\begin{cases} E_f = 0 : & F(x) = H(\bar{a}, \bar{q}; \alpha, \beta, \gamma, \bar{\delta}; \bar{x}) \\ E_f = 1 - \gamma : & F(x) = x^{1-\gamma} H(\bar{a}, \bar{q}_1; \alpha_1, \beta_1, \gamma_1, \bar{\delta}; \bar{x}) \end{cases} \quad (A7)$$

with

$$\bar{q} \equiv \bar{a}q \quad , \quad \bar{\delta} \equiv \alpha + \beta + 1 - \gamma - \delta \quad , \quad \bar{q}_1 \equiv \bar{a}q_1 \quad (A8)$$

B) The differential equation (26) appearing in the main text can be reduced to the Heun canonical form by means of the following change of function

$$f \longrightarrow F : \quad f(x) = x^k (x - 2)^l (x - b)^m F(x) \quad (A9)$$

A rather awkward but straightforward calculation shows that there are four sets of suitable values for the exponents (k, l, m) and consequently for the corresponding parameters $(a, q; \alpha, \beta, \gamma, \delta)$. Since we are looking for an analytical solution of equation (A1) in the neighbourhood of the origin ($x = 0$), we obtain eight possibilities, provided by the two Frobenius exponents $(0, 1 - \gamma)$ of the Heun equation at this point. Taking into account that the only correct Frobenius exponent from the starting equation (26) is $1/2$, we finally have only four possibilities. It can be checked that, naturally, all these four possibilities lead to the same analytical solution at $x = 0$, one of them being defined by the following set of parameters

$$E_f = 0 \quad \begin{cases} k = \frac{1}{2} \quad , \quad l = \frac{1}{2} \quad , \quad m = 0 \\ a = \frac{b}{2} \quad , \quad q = 3\frac{b}{2} + \frac{1}{2} \quad ; \quad \alpha = 4 \quad , \quad \beta = 1 \quad , \quad \gamma = \frac{5}{2} \quad , \quad \delta = \frac{5}{2} \end{cases} \quad (A10)$$

Now, since $|a| < 1$, we must resort to formulae (A6), (A8), which provide the following values for the parameters

$$\bar{a} = \frac{2}{b} \quad , \quad \bar{q} = 3 + \frac{1}{b} \quad , \quad \alpha = 4 \quad , \quad \beta = 1 \quad , \quad \gamma = \frac{5}{2} \quad , \quad \bar{\delta} = 1 \quad (A11)$$

which construct the solution shown in the main text (36).

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